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STEADY-STATE FRICTIONAL HEAT GENERATION ON AXISYMMETRIC SLIDING CONTACT OF A THERMOSENSITIVE SPHERE AND A FIXED, THERMALLY INSULATED BASE

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An approach to the construction of a calculation scheme for finding the contact area dimensions, pressure, and temperature for a thermosensitive sphere sliding along a fixed, thermally isolated base is suggested. The well-known solution for the constant thermal properties of the sphere material follows as a particular case of the solution obtained.

Introduction. The axisymmetric contact problem taking into account the steady-state heat release due to sliding friction was studied in [1–4]. In this case, a number of new (compared to the Hertz problem [5]) features in the behavior of the main contact characteristics was revealed. Thus, the limiting (critical) radius of the contact area with infinite increase in the pressing force was obtained numerically in [1] and analytically in [2]. The relation between this value and the onset of thermoelastic instability on the contact was found.

All the solutions mentioned above are obtained under the assumption of constant thermal properties of the interacting bodies. However, modern frictional materials operating at high contact pressures and temperatures are mostly thermosensitive, i.e., they have temperature-dependent thermal conductivities, specific heats, and linear thermal expansion coefficients. Methods of solution have mainly been developed for the temperature problems of the theory of friction [6–8]. The solutions of the corresponding problems of thermal elasticity are sparse [9], and for contact problems there are no solutions at all.

1. Formulation of the Problem. Let an elastic sphere of radius R be pressed by force P into a fixed base moving with velocity v. Because of friction in the contact area, heat is generated, which leads to formation of a temperature field and attendant thermal deformations. The latter have a substantial effect on the distribution of the contact pressure p.

It is assumed that:

(1) Vertical displacements of the sphere are independent of tangential stresses on the contact area, i.e., the friction affects the contact-area dimensions only through the generation of heat.

(2) The entire surface of the base and the free part of the sphere surface are thermally insulated. The intensity of the frictional heat flux q, directed to heating the sphere in the contact area, is proportional to the specific power of friction:

$$q = fvp \tag{1.1}$$

(f is the coefficient of friction). The time of heating of the sphere by the heat flux (1.1) is sufficient for the temperature field to reach a steady state;

(3) The material of the sphere is thermosensitive: its coefficients of heat transfer K and linear thermal expansion α are functions of the temperature T of the form

$$K(T) = K_0K^*(T), \qquad \alpha(T) = \alpha_0\alpha^*(T), \tag{1.2}$$

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where $K_0 = K(T_0)$, $\alpha_0 = \alpha(T_0)$, and $T_0 = 25°C$ is the temperature of the sphere points remote from the interaction region.

On the basis of assumptions 1 and 2 the problem will be considered in an axisymmetric formulation. We choose the origin of cylindrical coordinates $r\varphi z$ at the center of the contact circle of radius a directing the z axis inside the sphere. Assuming additionally that $a \ll R$, we formulate the boundary problems of the thermal conductivity and thermal elasticity for a sphere for a half-space, according to the Hertz approach [5].

2. The Thermal-Conductivity Problem. The temperature field T occurring as a result of heating of the thermosensitive body by the heat flux (1.1) is found from the solution of the nonlinear equation of thermal conductivity

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big[K(T)\rho\frac{\partial T}{\partial p}\Big] + \frac{\partial}{\partial \zeta}\Big[K(T)\frac{\partial T}{\partial \zeta}\Big] = 0 \quad (0 \leqslant \rho, \zeta < \infty), \tag{2.1}$$

subject to the boundary conditions

$$K\frac{\partial T}{\partial \zeta}\Big|_{\zeta=0} = -fvap(\rho)H(1-\rho), \qquad 0 \leqslant \rho < \infty; \tag{2.2}$$

$$T \to T_0, \quad \frac{\partial T}{\partial \rho} \to 0, \quad \frac{\partial T}{\partial \zeta} \to 0 \quad \text{for} \quad \sqrt{\rho^2+\zeta^2} \to \infty, \tag{2.3}$$

where $\rho = r/a$, $\zeta = z/a$, and $H(\cdot)$ is a Heaviside unit function.

Using the Kirchhoff transform

$$\Phi(T) \equiv \int_{T_0}^{T} K^*(T')\,dT' \tag{2.4}$$

in Eqs. (2.1) and conditions (2.2) and (2.3) yields

$$\nabla^2\Phi = 0, \qquad 0 \leqslant \rho, \zeta < \infty; \tag{2.5}$$

$$\frac{\partial \Phi}{\partial \zeta}\Big|_{\zeta=0} = -\frac{fva}{K_0}p(\rho)H(1-\rho), \qquad 0 \leqslant \rho < \infty; \tag{2.6}$$

$$\Phi \to 0, \quad \frac{\partial \Phi}{\partial \rho} \to 0, \quad \frac{\partial \Phi}{\partial \zeta} \to 0 \quad \text{for} \quad \sqrt{\rho^2+\zeta^2} \to \infty, \tag{2.7}$$

where $\nabla^2 = \partial^2/\partial\rho^2 + \rho^{-1}\partial/\partial\rho + \partial^2/\partial\zeta^2$ is the Laplacian.

The solution of Eq. (2.5) satisfying conditions (2.6) and (2.7) has the form [4]

$$\Phi(\rho,\zeta) = \frac{fva}{K_0}\int_0^\infty e^{-s\zeta}J_0(s\rho)\,ds\int_0^1 xp(x)J_0(xs)\,dx, \tag{2.8}$$

where $J_0(\cdot)$ is a zeroth-order Bessel function of the first kind.

We perform integration of [10]

$$\int_0^\infty e^{-s\zeta}J_0(s\rho)J_0(sx)\,ds = \frac{2}{\pi}\frac{K[\lambda(\rho,\zeta,x)]}{\sqrt{\zeta^2+(\rho+x)^2}}; \tag{2.9}$$

$$\frac{d}{d\zeta}\int_0^\infty e^{-s\zeta}J_0(s\rho)J_0(sx)\,ds = -\frac{2}{\pi}\frac{\zeta E[\lambda(\rho,\zeta,x)]}{\sqrt{\zeta^2+(\rho+x)^2}[\zeta^2+(\rho-x)^2]} \quad (\zeta > 0), \tag{2.10}$$

where $\lambda(\rho,\zeta,x) = 2\sqrt{\rho x/(\zeta^2+(\rho+x)^2)}$, $K(\cdot)$ and $E(\cdot)$ are complete elliptical integrals of the first and second kinds, respectively. Taking into account (2.9) and (2.10), for the function Φ (2.8) and its partial derivative

$\partial\Phi/\partial\zeta$ we obtain

$$\Phi(\rho,\zeta) = \frac{2fva}{\pi K_0} \int_0^1 xp(x) \frac{K[\lambda(\rho,\zeta,x)]}{\sqrt{\zeta^2+(\rho+x)^2}}\,dx; \tag{2.11}$$

$$\frac{\partial\Phi(\rho,\zeta)}{\partial\zeta} = -\frac{2fva}{\pi K_0} \int_0^1 xp(x) \frac{\zeta E[\lambda(\rho,\zeta,x)]}{\sqrt{\zeta^2+(\rho+x)^2}[\zeta^2+(\rho-x)^2]}\,dx \quad (\zeta > 0). \tag{2.12}$$

3. The Thermal-Elasticity Problem. Let us consider the equilibrium equations in displacements for a thermosensitive body [11]

$$\nabla^2 u_\rho - \frac{1}{\rho^2} u_\rho + \frac{1}{1-2\nu}\frac{\partial\theta}{\partial\rho} = \frac{2(1+\nu)}{1-2\nu}\alpha_0 \frac{\partial\Psi}{\partial\rho}, \quad \nabla^2 u_\zeta + \frac{1}{1-2\nu}\frac{\partial\theta}{\partial\zeta} = \frac{2(1+\nu)}{1-2\nu}\alpha_0\frac{\partial\Psi}{\partial\zeta}. \tag{3.1}$$

Here

$$\Psi(T) \equiv \int_{T_0}^{T} \alpha^*(T')\,dT', \tag{3.2}$$

$\theta = \partial u_\rho/\partial\rho + u/\rho + \partial u_\zeta/\partial\zeta$, $u_\rho = u_r/a$, $u_\zeta = u_z/a$, u_r and u_z are the components of the displacement vector in the r and the z axis directions, and ν is Poisson's coefficient.

The partial solution of the thermal-elasticity equations (3.1) is written as [12]

$$u_\rho^{(1)} = \frac{\partial F}{\partial\rho}, \qquad u_\zeta^{(1)} = \frac{\partial F}{\partial\zeta}, \tag{3.3}$$

where the thermal elastic potential F and the function Ψ (3.2) satisfy the equation

$$\nabla^2 F = \beta\Psi \qquad [\beta = \alpha_0(1+\nu)/(1-\nu)]. \tag{3.4}$$

The stresses $\sigma_{\zeta\zeta}^{(1)}$ and $\sigma_{\rho\zeta}^{(1)}$ corresponding to the function F are determined from the formulas [12]

$$\sigma_{\zeta\zeta}^{(1)} = -2\mu\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F}{\partial\rho}\right), \qquad \sigma_{\rho\zeta}^{(1)} = 2\mu\frac{\partial^2 F}{\partial\rho\,\partial\zeta} \tag{3.5}$$

(μ is the displacement modulus).

The solution of Eq. (3.4) satisfying the descending conditions as $\sqrt{\rho^2+\zeta^2} \to \infty$ has the form

$$F(\rho,\zeta) = \beta\int_0^\infty J_0(s\rho)\,ds \int_\zeta^\infty \bar{\Psi}(s,y)\sinh[s(y-\zeta)]\,dy, \tag{3.6}$$

where

$$\bar{\Psi}(s,y) = \int_0^\infty \rho\Psi(\rho,y)J_0(s\rho)\,d\rho. \tag{3.7}$$

Substituting the function F (3.6) into relations (3.5), we find

$$\sigma_{\zeta\zeta}^{(1)} = 2\mu\beta\int_0^\infty s^2 J_0(s\rho)\,ds\int_0^\infty \bar{\Psi}(s,y)\sinh(sy)\,dy, \quad \sigma_{\rho\zeta}^{(1)} = 2\mu\beta\int_0^\infty s^2 J_1(s\rho)\,ds\int_0^\infty \bar{\Psi}(s,y)\cosh(sy)\,dy \tag{3.8}$$

for $\zeta = 0$, i.e., the half-space surface is not free. Therefore, let us consider the additional problem of the effect on the surface $\zeta = 0$ of the forces initiating stresses (denoted by subscript 2) such that

$$\sigma_{\zeta\zeta}^{(1)} + \sigma_{\zeta\zeta}^{(2)} = 0, \quad \sigma_{\rho\zeta}^{(1)} + \sigma_{\rho\zeta}^{(2)} = 0 \quad \text{for} \quad \zeta = 0. \tag{3.9}$$

The stress state caused by the forces distributed axisymmetrically over the surface of the elastic half-space is defined by the Love function L [13] from the solution of the biharmonic equation

$$\nabla^2\nabla^2 L = 0. \tag{3.10}$$

The vertical displacements $u_\zeta^{(2)}$ and the stresses $\sigma_{\zeta\zeta}^{(2)}$ and $\sigma_{\rho\zeta}^{(2)}$ are related to the function L by

$$u_\zeta^{(2)} = \frac{1}{1-2\nu}\left[2(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial\zeta^2}\right], \quad \sigma_{\zeta\zeta}^{(2)} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial\zeta}\left[(2-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial\zeta^2}\right],$$
$$\sigma_{\rho\zeta}^{(2)} = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial\rho}\left[(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial\zeta^2}\right]. \tag{3.11}$$

The general solution of Eq. (3.10) decreasing as $\rho \to \infty$ or $\zeta \to \infty$ has the form

$$L = \int\limits_0^\infty s[A_1(s) + \zeta A_2(s)]\,\mathrm{e}^{-s\zeta} J_0(s\rho)\,ds. \tag{3.12}$$

Substituting L (3.12) into Eqs. (3.11), we obtain

$$u_\zeta^{(2)} = -\frac{1}{1-2\nu}\int\limits_0^\infty s^2\{s[A_1(s)+\zeta A_2(s)] + 2(1-2\nu)A_2(s)\}\,\mathrm{e}^{-s\zeta}J_0(s\rho)\,ds; \tag{3.13}$$

$$\sigma_{\zeta\zeta}^{(2)} = \frac{2\mu}{1-2\nu}\int\limits_0^\infty s^3\{s[A_1(s)+\zeta A_2(s)] + (1-2\nu)A_2(s)\}\,\mathrm{e}^{-s\zeta}J_0(s\rho)\,ds; \tag{3.14}$$

$$\sigma_{\rho\zeta}^{(2)} = \frac{2\mu}{1-2\nu}\int\limits_0^\infty s^3\{s[A_1(s)+\zeta A_2(s)] - 2\nu A_2(s)\}\,\mathrm{e}^{-s\zeta}J_1(s\rho)\,ds. \tag{3.15}$$

From boundary conditions (3.9), taking into account formulas (3.8), (3.14), and (3.15), we find

$$A_1(s) = -(1-2\nu)\,\beta s^{-2}\left[(1-2\nu)\int\limits_0^\infty \bar\Psi(s,\zeta)\cosh(s\zeta)\,d\zeta + 2\nu\int\limits_0^\infty \bar\Psi(s,\zeta)\sinh(s\zeta)\,d\zeta\right],$$

$$A_2(s) = (1-2\nu)\,\beta s^{-1}\int\limits_0^\infty \bar\Psi(s,\zeta)\,\mathrm{e}^{-s\zeta}\,d\zeta.$$

Then, from relations (3.3), (3.6), and (3.13) we obtain the following expression for the vertical displacements of the thermosensitive half-space surface heated by the frictional heat flux (1.1):

$$u_\zeta^{\mathrm{th}}(\rho) \equiv u_\zeta^{(1)}(\rho,0) + u_\zeta^{(2)}(\rho,0) = -2(1+\nu)\alpha_0\int\limits_0^\infty sJ_0(s\rho)\,ds\int\limits_0^\infty \bar\Psi(s,\zeta)\,\mathrm{e}^{-s\zeta}\,d\zeta. \tag{3.16}$$

Substituting the expression for the transformant $\bar\Psi$ (3.7) into Eq. (3.16) and taking into account (3.2), we obtain

$$u_\zeta^{\mathrm{th}}(\rho) = -2(1+\nu)\alpha_0\int\limits_0^\infty sJ_0(s\rho)\,ds\int\limits_0^\infty \mathrm{e}^{-s\zeta}\,d\zeta\int\limits_0^\infty xJ_0(sx)\,dx\int\limits_{T_0}^{T}\alpha^*(T')\,dT'. \tag{3.17}$$

Since

$$\int\limits_{T_0}^{T}\alpha^*(T')\,dT' = \Phi(T) + \int\limits_{T_0}^{T}[\alpha^*(T') - K^*(T')]\,dT',$$

where the function Φ is given by (2.8), equality (3.17) is rewritten in the form

$$u_\zeta^{\mathrm{th}}(\rho) = -\delta f v a\, u_{\zeta 1}^{\mathrm{th}}(\rho) - 2(1+\nu)\alpha_0 u_{\zeta 2}^{\mathrm{th}}(\rho), \qquad \rho \geqslant 0. \tag{3.18}$$

Here

$$u_{\zeta 1}^{\rm th}(\rho) = \int\limits_0^\infty s^{-1} J_0(s\rho)\, ds \int\limits_0^1 x p(x) J_0(sx)\, dx, \tag{3.19}$$

$$u_{\zeta 2}^{\rm th}(\rho) = \int\limits_0^\infty s J_0(s\rho)\, ds \int\limits_0^\infty e^{-s\zeta}\, d\zeta \int\limits_0^\infty x J_0(sx)\, dx \int\limits_{T_0}^{T} [\alpha^*(T') - K^*(T')]\, dT', \tag{3.20}$$

and $\delta = \alpha_0(1+\nu)/K_0$ is the coefficient of thermal deformation [1].

The first term in formula (3.18) determines the solution of the thermal-elasticity boundary-layer problem considered for constant (temperature-independent) properties of the half-space material [2]. Below, this problem will be called linear. The second term in formulas (3.18) introduces a correction due to the thermosensitivity of the material.

From (3.20) it follows that when the equality $\alpha^*(T) = K^*(T)$ is satisfied, the vertical displacement $u_{\zeta 2}^{\rm th}(\rho) = 0$ and the expression for $u_\zeta^{\rm th}(\rho)$ will be the same as that in the corresponding linear problem. For $\alpha^*(T) > K^*(T)$, the thermal distortion of the body surface increases compared to the linear case, and for $\alpha^*(T) < K^*(T)$, it decreases.

We transform relation (3.20) to a more convenient form for applications. Integration by parts yields

$$u_{\zeta 2}^{\rm th}(\rho) = \int\limits_0^\infty J_0(s\rho)\, ds \int\limits_0^\infty x J_0(sx)\, dx \int\limits_{T_0}^{T} [\alpha^*(T') - K^*(T')]\, dT' \Big|_{\zeta=0}$$

$$+ \int\limits_0^\infty J_0(s\rho)\, ds \int\limits_0^\infty e^{-s\zeta} d\zeta \int\limits_0^\infty x J_0(sx) \frac{\partial \Phi}{\partial \zeta} \frac{\alpha^*(T) - K^*(T)}{K^*(T)}\, dx. \tag{3.21}$$

Taking into account the value of integral (2.9), from (3.21) we find

$$u_{\zeta 2}^{\rm th}(\rho) = \frac{2}{\pi}\Big\{ \int\limits_0^\infty x \int\limits_{T_0}^{T(x,0)} [\alpha^*(T') - K^*(T')]\, dT' \Big|_{\zeta=0} \frac{K[\lambda(\rho,0,x)]}{\rho + x}\, dx$$

$$+ \int\limits_0^\infty d\zeta \int\limits_0^\infty x \frac{K[\lambda(\rho,\zeta,x)]}{\sqrt{\zeta^2 + (\rho+x)^2}} \frac{\partial \Phi}{\partial \zeta} \frac{\alpha^*(T) - K^*(T)}{K^*(T)}\, dx \Big\}, \tag{3.22}$$

where $\partial\Phi/\partial\zeta$ is defined by formula (2.12).

4. The Contact Problem. Write the condition of contact between the sphere and the fixed base as:

$$u_\zeta^{\rm e}(\rho) + u_\zeta^{\rm th}(\rho) = \Delta - \frac{a}{2R}\rho^2 \qquad (0 \leqslant \rho \leqslant 1). \tag{4.1}$$

Here $u_\zeta^{\rm e} = u_z^{\rm e}/a$ is the dimensionless normal displacement of the sphere points caused by the mechanical load, $u_\zeta^{\rm th}$ is the thermal deformation due to frictional heat generation [defined by relations (3.18), (3.19), and (3.22)], and Δ is the vertical displacement of the remote point on the sphere. The displacement $u_\zeta^{\rm e}$ has the form [14]

$$u_\zeta^{\rm e}(\rho) = \frac{2(1-\nu)}{\pi\mu} \int\limits_0^1 x p(x) \frac{K[\lambda(\rho,0,x)]}{\rho + x}\, dx, \qquad \rho \geqslant 0. \tag{4.2}$$

Substituting the displacements $u_\zeta^{\rm e}$ (4.2) and $u_\zeta^{\rm th}$ (3.18) into condition (4.1) yields the nonlinear integral equation with respect to the contact pressure $p(\rho)$. To determine Δ we use the condition of equilibrium.

$$\int\limits_0^1 \rho p(\rho)\, d\rho = \frac{P}{2\pi a^2}. \tag{4.3}$$

Finding the exact solution to the nonlinear integral equation is a cumbersome task, complicated by the fact that the radius of the contact area a is not known a priori. Therefore, in [15, 16], we propose the following technique for constructing an approximate solution of the contact problems. The contact pressure is written in the form

$$p(\rho) = p_0\{d + (5/4)(3-2d)\rho^2\}\sqrt{1-\rho^2}, \quad 0 \leqslant \rho \leqslant 1, \quad p_0 = P/(\pi a^2), \tag{4.4}$$

where the unknown parameter is $d = p(0)/p_0$. We note that the pressure distribution (4.4) exactly satisfies the equilibrium condition (4.3).

Substituting $p(\rho)$ (4.4) into (4.2) and integrating, we find

$$u_\zeta^e(\rho) = \frac{P}{a^2}\gamma\Big\{\frac{1}{16}(15-6d)\Big(1-\frac{1}{2}\rho^2\Big) - \frac{15}{64}(3-2d)\Big(1-\rho^2+\frac{3}{8}\rho^4\Big)\Big\}, \tag{4.5}$$

where $\gamma = (1-\nu)/\mu$.

The normal displacement of the sphere surface due to frictional heating is approximated by the fourth-order polynomial

$$u_\zeta^{\rm th}(\rho) \approx \tilde{u}_\zeta^{\rm th}(\rho) = u_\zeta^{\rm th}(0) + C_1\rho^2 + C_2\rho^4. \tag{4.6}$$

The coefficients C_i $(i = 1, 2)$ are determined from the physical conditions

$$\tilde{u}_\zeta^{\rm th}(0) - \tilde{u}_\zeta^{\rm th}(1) = U_1, \quad \int_0^1 [\tilde{u}_\zeta^{\rm th}(\rho) - \tilde{u}_\zeta^{\rm th}(1)]\rho\, d\rho = U_2, \tag{4.7}$$

where

$$U_i = -\frac{\delta f v P}{\pi a}U_i^{(1)} - 2(1+\nu)\alpha_0 U_i^{(2)}, \qquad i = 1, 2,$$

$$U_1^{(j)} = V_j[u_{\zeta j}^{\rm th}(0) - u_{\zeta j}^{\rm th}(1)], \quad U_2^{(j)} = V_j\int_0^1 [u_{\zeta j}^{\rm th}(\rho) - u_{\zeta j}^{\rm th}(1)]\,\rho d\rho, \quad j = 1, 2, \tag{4.8}$$

$$V_1 = \pi a^2/P, \qquad V_2 = 1,$$

and $u_{\zeta j}^{\rm th}$ $(j = 1, 2)$ is found from formulas (3.19) and (3.22). Substituting $p(\rho)$ from (4.4) into relations (3.19), we obtain

$$U_1^{(1)} = \frac{1}{3}\Big(\frac{4}{3} - \ln 2\Big)d + \frac{1}{6}\Big(\frac{31}{30} - \ln 2\Big)(3-2d), \quad U_2^{(1)} = \frac{d}{20} + \frac{1}{56}(3-2d).$$

Taking into account formulas (2.4), (2.11), (2.12), (3.19), (3.22), and (4.4), from relations (4.6)–(48) we have

$$C_i = -\frac{\delta f v P}{\pi a}C_i^{(1)}(d) - 2(1+\nu)\alpha_0 C_i^{(2)}\Big(\frac{fvP}{aK_0}, d\Big), \qquad i = 1, 2, \tag{4.9}$$

where $C_1^{(j)} = 3U_1^{(j)} - 12U_2^{(j)}$ and $C_2^{(j)} = -4U_1^{(j)} + 12\,U_2^{(j)}$ $(j = 1, 2)$.

Substituting u_ζ^e (4.5) and $u_\zeta^{\rm th}$ (4.6) with allowance for (4.9) into condition (4.1) and comparing the coefficients at the same powers of ρ, we obtain

$$\begin{aligned} &\frac{9d}{32} - \frac{15}{64} + \frac{\delta f v a}{\pi\gamma}C_1^{(1)}(d) + \frac{2(1+\nu)\alpha_0 a^2}{\gamma P}C_1^{(2)}\Big(\frac{fvP}{aK_0}, d\Big) = \frac{a^3}{2PR\gamma}, \\ &\frac{45}{512}(3-2d) + \frac{\delta f v a}{\pi\gamma}C_2^{(1)}(d) + \frac{2(1+\nu)\alpha_0 a^2}{\gamma P}C_2^{(2)}\Big(\frac{fvP}{aK_0}, d\Big) = 0. \end{aligned} \tag{4.10}$$

Denote

$$a_0 = a/a_{\rm H}, \qquad a_1 = (1+\nu)\alpha_0 a_{\rm H}^2/\gamma P, \qquad a_2 = fvP/a_{\rm H}K_0, \tag{4.11}$$

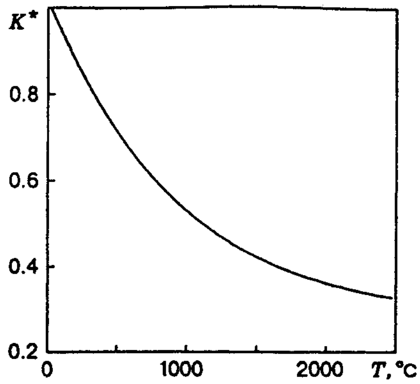

Fig. 1

TABLE 1

i	k_i	φ_i
0	1.018 86	25.0
1	$-7.629\,80 \cdot 10^{-4}$	1.005 39
2	$3.445\,10 \cdot 10^{-7}$	$3.617\,45 \cdot 10^{-4}$
3	$-7.768\,78 \cdot 10^{-11}$	$1.108\,48 \cdot 10^{-7}$
4	$7.036\,91 \cdot 10^{-15}$	$2.411\,02 \cdot 10^{-10}$
5	—	$-1.098\,51 \cdot 10^{-13}$

where $a_{\rm H}^3 = 3PR\gamma/8$ is the radius of the contact area in the Hertz problem [5]. Then Eqs. (4.10) are written as

$$\frac{9d}{32} - \frac{15}{64} + \frac{1}{\pi} a_0 a_1 a_2 C_1^{(1)}(d) + 2a_0^2 a_1 C_1^{(2)}\left(\frac{a_2}{a_0}, d\right) = \frac{3}{16} a_0^3,$$
$$\frac{45}{512}(3-2d) + \frac{1}{\pi} a_0 a_1 a_2 C_2^{(1)}(d) + 2a_0^2 a_1 C_2^{(2)}\left(\frac{a_2}{a_0}, d\right) = 0. \tag{4.12}$$

Thus, we arrive at a system of two nonlinear equations in a_0 and d. The input parameters of the problem are the quantities a_1 and a_2 (4.11), which have dimensions [°C^{-1}] and [°C], respectively. We note that the product $a_1 a_2$ is the only dimensionless input parameter of the corresponding linear problem [1, 2].

For calculations, it is convenient to specify the parameters a_0 and a_2 and assume that the quantities a_1 and d are to be sought. Then, the second equation of (4.12) leads to

$$a_1 = \frac{45(2d-3)/512}{(1/\pi) a_0 a_2 C_2^{(1)}(d) + 2a_0^2 C_2^{(2)}(a_2/a_0, d)}. \tag{4.13}$$

Substitution of a_1 (4.13) into the first equation of (4.12) yields one nonlinear equation for d.

We note that setting $C_i^{(2)} = 0$ ($i = 1, 2$) in Eqs. (4.12), we obtain a solution of the corresponding linear problem [1] for which $d = 1.91$ and $a_0 a_1 a_2 = 2.01$ as $P \to \infty$. The exact value of the limiting (as $P \to \infty$) radius of the contact area in the linear problem was found in [2]: $a_0^* = 2/(a_1 a_2)$. Thus, the approximation of the contact pressure in the form (4.4) gives an approximate solution that differs slightly from the exact solution.

5. Numerical Analysis. We assume that the material of the thermosensitive sphere is graphite, for which $K_0 = 79.8$ W/(m · K). The temperature dependence of the function K^* (1.2) obtained in [17] is shown in Fig. 1. Approximation of these results by the least-square method with an absolute error not exceeding 3% has the form

$$K^*(T) = \sum_{i=0}^{4} k_i T^i$$

(the coefficients k_i are given in Table 1). The linear thermal expansion coefficient $\alpha(T)$ was assumed to be constant [$\alpha^*(T) = 1$].

To determine the temperature T from the Kirchhoff variable Φ (2.4) we obtain the polynomial (with an absolute error not exceeding 1%) dependence

$$T(\Phi) = \sum_{i=0}^{5} \varphi_i \Phi^i.$$

The values of the coefficients φ_i are also given in Table 1.

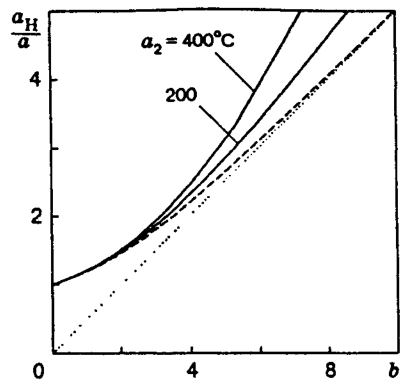

Fig. 2

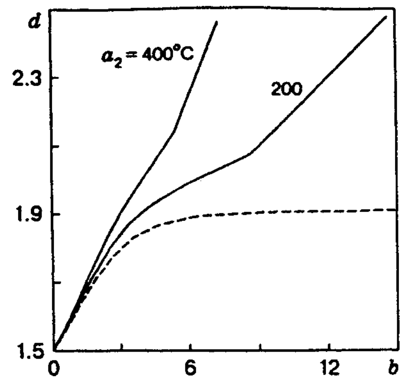

Fig. 3

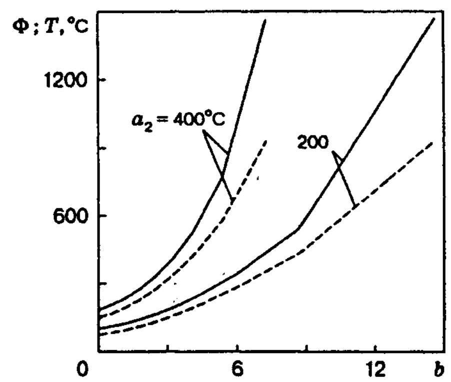

Fig. 4

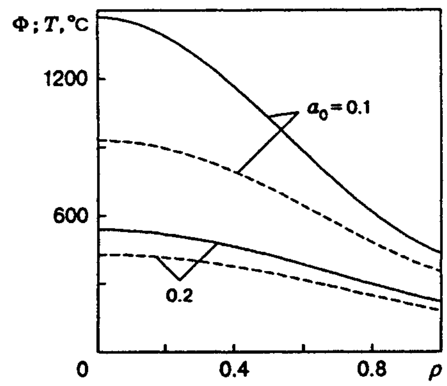

Fig. 5

The dependence of the quantity $a_0^{-1} = a_H/a$, which is the reverse of the dimensionless radius of the contact area a_0 (4.11), on the parameter $b = a_1 a_2$ for two values of a_2 (4.11) is shown in Fig. 2. At $b = 0$, there is no frictional heating: $a = a_H$. In the linear problem (the dashed curve), an increase in the parameter b leads to a decrease in the contact area and makes the radius tend to its critical value $a_0^* = 2/b$ (the dotted curve). It can be seen that even for $b > 6$, the actual value of the contact-area radius can be replaced by the limiting value a_0^*. The influence of the thermosensitivity on the size of the contact area is manifested primarily in the fact that for a fixed input parameter b, the contact-circle radius is substantially smaller than that in the linear problem. This difference increases with increase in b or a_2 for a fixed b. In the nonlinear version of the problem, the contact-area radius does not tend to a certain limiting value within the range of input parameters considered.

The quantity d (the ratio of the pressure at the center of the contact area to its mean value) as a function of the parameter b for two values of a_2 is shown in Fig. 3. In the linear problem (the dashed curve), an increase in b leads to an increase in d and for $b > 6$, one can set $d = 1.91$. In the case of a thermosensitive material, d increases with b but does not reach the limiting value. For fixed b, the value of d is the larger, the larger the parameter a_2. Thus, the influence of thermosensitivity shows up as an increase in the maximum contact pressure and localization of contact stresses in the central part of the contact area.

The temperature T as a function of the parameter b (the solid curves) for $a_2 = 200$ and 400°C at the center of the contact circle ($\rho = 0$, $\zeta = 0$) is shown in Fig. 4. The dashed curves correspond to values of the Kirchhoff function Φ (2.11). As should be expected, a decrease in the radius of the contact area and the increase in pressure in its central part lead to a sharp rise in temperature in the central part of the contact circle.

The distribution of the surface contact temperature (the solid curves) at $a_2 = 200°C$ for two values of a_0 is shown in Fig. 5. The dashed curves show the distribution of the Kirchhoff function Φ (2.11). The maximum temperature is attained at the center of the area, decreasing rapidly with distance ρ. In this case, the thermosensitivity effect decreases sharply with distance from the center of the contact area.

The analysis performed showed that the influence of thermosensitivity on the contact characteristics should be taken into account at temperatures of about 500°C and higher. We note that the effect of the heat-generation power fvP (the parameter a_2) on the same characteristics is significant compared to that in the linear problem.

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